Complex Numbers

- **Definition :~** A complex number z is defined to be an ordered pair of real numbers (a,b) that satisfies the following condition (i) and following laws of operations (ii) & (iii)

  i) $(a,b) = (c,d)$ iff $a=c,\ b=d$

  ii) $(a,b)+(c,d) = (a+c,\ b+d)$

  iii) $(a,b).(c,d) = (ac-bd,\ ad+bc)$

→ of the ordered pair (a,b), a is called real part of z and b is called imaginary part of z, denoted by $\text{Re}\,z$ and $\text{Im}\,z$ respectively.

→ i) $z_1+z_2 = z_2+z_1$; ii) $z_1 \cdot z_2 = z_2 \cdot z_1$ iii) $z_1(z_2+z_3) = z_1z_2 + z_2z_3$

- **Conjugate of a complex number :~**

  Let $z = a+bi$ be a complex number. The conjugate of z, denoted by $\bar{z}$ defined by $a-bi$.

→ Properties :- i) $\bar{\bar{z}} = z$

ii) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

iii) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

iv) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

vi) $z+\bar{z} = 2.\text{Re}\,z$, $z-\bar{z} = 2i\,\text{Im}\,z$

vii) $z.\bar{z}$ is purely real

- **Modulus of a complex number**

  Let $z = a+bi$ be a complex number. The non-negative real numbers $\sqrt{a^2+b^2}$ is said to be the absolute value or the modulus of z, denoted by $|z|$ / mod z / mod $(a+bi)$.

→ i) $|z_1z_2| = |z_1||z_2|$

ii) $|z| \geq \text{Re}\,z$, $|z| \geq \text{Im}\,z$,

iii) $|z_1+z_2| \leq |z_1|+|z_2|$

- **Polar form :~** A nonzero complex number $z = a+ib$ is represented in the form $z = r(\cos\theta + i\sin\theta)$, is called the polar form / modulus-amplitude form.

→ i) r is the distance of the point (a,b) from the origin
ii) r is said to be modulus of z
iii) For every point except origin, $r>0$
iv) For the pt. $(0,0)$, $r=0$

→ i) θ is the angle made by the radius vector through the pt (a,b) with real axis.
ii) θ is called an argument or amplitude of z.
iii) θ can't determined for $z=0$
iv) $z \neq 0$, θ has infinitely many values differing from one another by a multiple of 2π.
v) All values of θ are expressed as $\mathrm{Arg}\,z$ (or $\mathrm{Amp}\,z$).
If α be a value of θ satisfying the relation $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$, then $\boxed{\mathrm{Arg}\,z = \alpha + 2n\pi,\ n \in \mathbb{Z}.}$
vi) The principal argument of z, denoted by $\arg z$, is defined by θ which satisfy $-\pi < \theta < \pi$

→ $z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{a^2+b^2}$
θ is determined from $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$, not from the single relation $\theta = \tan^{-1}\frac{b}{a}$.

→ If z_1, z_2 are two non zero complex number then
i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi$ where
$k=0$ if $-\pi < \arg z_1 + \arg z_2 \leq \pi$
$k=1$ if $\arg z_1 + \arg z_2 \leq -\pi$
$k=-1$ if $\arg z_1 + \arg z_2 > \pi$

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$ where
$k=0$ if $-\pi < \arg z_1 - \arg z_2 \leq \pi$
$k=1$ if $\arg z_1 - \arg z_2 \leq -\pi$
$k=-1$ if $\arg z_1 - \arg z_2 > \pi$

Exercises-2A

(19) i) $z = \frac{1+\sqrt{3}i}{1+i}$

Now $z = \frac{z_1}{z_2}$ where $z_1 = 1+\sqrt{3}i$, $z_2 = 1+i$

$\therefore |z| = \left|\frac{1+\sqrt{3}i}{1+i}\right| = \frac{|1+\sqrt{3}i|}{|1+i|} = \frac{\sqrt{1+3}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Now $z_1 = 1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \Rightarrow \arg z_1 = \frac{\pi}{3}$

$z_2 = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow \arg z_2 = \frac{\pi}{4}$

We know that $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 + 2k\pi$

Now $\arg z_1 - \arg z_2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \in (-\pi, \pi] \Rightarrow k=0$

$\therefore \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 = \frac{\pi}{12}$ i.e. $\arg\left(\frac{1+\sqrt{3}i}{1+i}\right) = \frac{\pi}{12}$

ii) $\boxed{z = \frac{-1+i}{1-\sqrt{3}i}}$

Let $z = \frac{z_1}{z_2}$ where $z_1 = -1+i$ and $z_2 = 1-\sqrt{3}i$

Now $z_1 = -1+i = \sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left\{\cos\left(\pi-\frac{\pi}{4}\right) + i\sin\left(\pi-\frac{\pi}{4}\right)\right\}$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore |z_1| = \sqrt{2}$, $\arg z_1 = \frac{3\pi}{4}$

And $z_2 = 1-\sqrt{3}i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$

$\therefore |z_2| = 2$, $\arg z_2 = -\frac{\pi}{3}$

Now $\arg z_1 - \arg z_2 = \frac{3\pi}{4} + \frac{\pi}{3} \neq \frac{13\pi}{12} > \pi$ so $k = -1$

$\therefore \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 - 2\pi = \frac{13\pi}{12} - 2\pi = -\frac{11\pi}{12}$

iii) $\boxed{z = 1+i\tan\theta;\ \frac{\pi}{2} < \theta < \pi}$

$\therefore$ ~~$|z| = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$~~

$r^2 = 1+\tan^2\theta = \sec^2\theta$

$\Rightarrow r = \pm\sec\theta$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$

Since $r > 0 \Rightarrow r = -\sec\theta > 0\ \forall\ \theta \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \mathrm{mod}\, z = -\sec\theta$

again, $z = 1 + i\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta + i\sin\theta}{\cos\theta}$

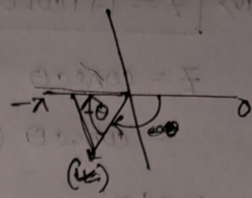
$\Rightarrow z = \sec\theta(\cos\theta + i\sin\theta)$

$\Rightarrow z = -\sec\theta(-\cos\theta - i\sin\theta)$

$= -\sec\theta\left(\cos(-\pi+\theta) + i\sin(-\pi+\theta)\right)$

$\therefore \arg z = -\pi+\theta$

(19) vii) $\boxed{z = 1+\cos 2\theta + i\sin 2\theta, \quad \frac{\pi}{2} < \theta < \pi}$



$z = 2\cos^2\theta + 2i\sin\theta\cos\theta$

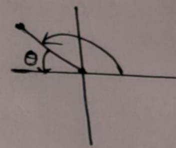
$= 2\cos\theta\,(\cos\theta + i\sin\theta)$

$= -2\cos\theta\,(-\cos\theta - i\sin\theta)$

$= -2\cos\theta\,(\cos(-\pi+\theta) + i\sin(-\pi+\theta))$, $\mod z = -2\cos\theta$

$\arg z = -\pi+\theta$

(19) viii) $\boxed{z = 1+\cos 2\theta - i\sin 2\theta, \quad +\frac{\pi}{2} < \theta < \pi}$



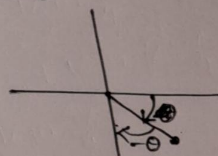
$\therefore z = 2\cos^2\theta - 2i\sin\theta\cos\theta = 2\cos\theta\,(\cos\theta - i\sin\theta)$

$= -2\cos\theta\,(-\cos\theta + i\sin\theta)$

$= -2\cos\theta\,(\cos(\pi-\theta) + i\sin(\pi-\theta))$

$\therefore \text{Mod } z = -2\cos\theta, \quad \arg z = \pi-\theta$

(19) ix) $\boxed{z = 1-\cos\theta\,(\cos\theta + i\sin\theta), \quad 0 < \theta < \pi}$



$\therefore z = 1-\cos^2\theta - i\cos\theta\sin\theta = \sin^2\theta - i\cos\theta\sin\theta$

$= \sin\theta\,(\sin\theta - i\cos\theta)$

$= \sin\theta\left\{\cos\left(-\frac{\pi}{2}+\theta\right) + i\sin\left(-\frac{\pi}{2}+\theta\right)\right\}$

$\therefore \mod z = \sin\theta, \quad \arg z = -\frac{\pi}{2}+\theta$

(19) x) $\boxed{z = 1-\sin\theta\,(\sin\theta + i\cos\theta), \quad \frac{\pi}{2} < \theta < \pi}$

$\therefore z = 1-\sin^2\theta - i\sin\theta\cos\theta = \cos^2\theta - i\sin\theta\cos\theta$

$= \cos\theta\,(\cos\theta - i\sin\theta)$

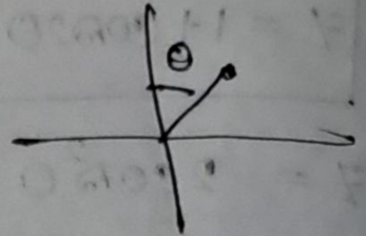
$= -\cos\theta\,(-\cos\theta + i\sin\theta)$

$= -\cos\theta\,(\cos(\pi-\theta) + i\sin(\pi-\theta))$

$\therefore \text{Mod } z = -\cos\theta$

$\arg z = \pi-\theta$

(19) iv) $\boxed{z = 1 + i\cot\theta, \ 0 < \theta < \pi}$



$$z = \operatorname{cosec}\theta\,(\sin\theta + i\cos\theta)$$

$$= \operatorname{cosec}\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right)$$

Mod $z = \operatorname{cosec}\theta$, $\arg z = \frac{\pi}{2} - \theta$

(20) Given $|z|=1$, $\text{amp}\, z=\theta$ $(0<\theta<\pi)$

let $z=r(\cos\theta+i\sin\theta)$, since $r=|z|=1 \Rightarrow z=\cos\theta+i\sin\theta$

i) $z_1=\dfrac{1-z}{1+z}=\dfrac{1-\cos\theta-i\sin\theta}{1+\cos\theta+i\sin\theta}=\dfrac{2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}$

$=\tan\frac{\theta}{2}\cdot\left(\dfrac{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}\right)=\tan\frac{\theta}{2}\cdot\left\{\dfrac{\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}-\left(i\sin\frac{\theta}{2}\right)^2}\right\}$

$=\tan\frac{\theta}{2}\cdot\left(\dfrac{\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}-i\sin^2\frac{\theta}{2}-i\cos^2\frac{\theta}{2}-\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}\right)$

$=\tan\frac{\theta}{2}(-i)=\tan\frac{\theta}{2}\cdot(0-1.i)=\tan\frac{\theta}{2}\left\{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right\}$

$\Rightarrow \text{Mod}\, z_1=\tan\frac{\theta}{2}$, $\arg z_1=-\pi/2$